A Hamiltonian for the description of a non-relativistic spin-1/2 free particle

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Abstract

We propose a Hamiltonian for a nonrelativistic spin 1/2 *free* particle (e.g. an electron) and find that it contains information of its internal degrees of freedom in the rest coordinate system. We comment on the dynamical symmetry associated with the electron *Zitterbewegung*.

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The derivation of the Dirac equation [1] begins with the attempt to linearize the Klein-Gordon equation for the free particle, which is quadratic in the energy-momentum operator $\widehat{p}^{\mu} = i\hbar\partial/\partial q_{\mu}$ (we use the metric signature $g(+---)$), i.e., $\left((c\widehat{p}^{0})^{2} - c^{2}\widehat{\mathbf{p}}^{2} - m_0^2 c^4\right)\Phi(q) = 0$, where c is the velocity of light, m_0 is the mass of the particle, $q = (q^0_, q^i) = (ct, \mathbf{q})$, and Φ a Lorentz scalar wave function of q. It has a $(c\widehat{p}^{0})^{2} = -(\hbar c)^2 \partial^2/\partial q_0^2$ term: this leads to a continuity equation with a probability density containing $\partial/\partial q_0$, and hence to negative probability. To solve these problems we require an equation linear in $\widehat{p} = (\widehat{p}^0, \widehat{p}^i) = (E/c, \widehat{\mathbf{p}})$. In fact, a Lorentz invariant wave equation can be constructed [1] in the form $(\widehat{p}^0 - \alpha{\cdot}\widehat{\mathbf{p}} - m_0 c\beta)\,\Psi(q) = 0$. The operator in this equation contains q_μ , $\mu = 0, 1, 2, 3$, only in derivative $\partial/\partial q_\mu$. Notice that this implies the existence of a reference system with a well defined *origin*, where the wave equation has been written down. On the other hand, since "all space-time points are equivalent", α and β "must not involve q" [1].

It is well known [2] that in nonrelativistic quantum mechanics it is possible to construct a Dirac-like *wave equation* to describe a spin 1/2 particle: the Lévy-Leblond equation (LE). However, from the LE it is not possible to define a Hamiltonian for the description of a given quantum system since the time derivative in the wave function is multiplied by a *singular* 4×4 matrix. On the other hand we know that the Pauli equation does not describe *Zitterbewegung*.

In this article we want to partially remedy these problems by proposing a Hamiltonian for a spin 1/2 particle (e.g. an electron) which both it is intrinsically nonrelativistic and contains information of its spin in the rest system. The "price" we have to pay is that we need to incorporate a quadratic term in $\widehat{\mathbf{p}}$ in the Hamiltonian. As this is preliminary work, we shall concern ourselves only with the case of the free particle.

To begin with, we consider the classical expression for the energy of a nonrelativistic free particle, including its rest energy:

$$E(\mathbf{p}) = m_0 c^2 + \frac{\mathbf{p}^2}{2m_0}. \tag{1}$$

If we square this equation we obtain

$$E^2(\mathbf{p}) = m_0^2c^4 + c^2\mathbf{p}^2 + \frac{\left(\mathbf{p}^2\right)^2}{4m_0^2}. \tag{2}$$

We observe that the first two terms of Eq.(2) (taking them together) resemble the expression for the squared of the free particle Dirac equation (the Klein-Gordon equation). In quantum mechanics, the "square root" of (2), for a *strictly* nonrelativistic spin 1/2 particle, can be written now as the Hamiltonian operator

$$\widehat{H} = c\alpha\cdot\widehat{\mathbf{p}} + m_0c^2\beta - i\beta\gamma_5\frac{(\alpha\cdot\widehat{\mathbf{p}})^2}{2m_0}, \tag{3}$$

where α_i, β are the Dirac matrices, and $\gamma_5 \equiv \gamma_0\gamma_1\gamma_2\gamma_3$ in the Dirac representation [3]. Thus in the process of linearizing Eq.(2), we pick up a quadratic term in the momentum $\widehat{\mathbf{p}}$ in Eq.(3).

The corresponding Schrödinger equation is written as usually:

$$\widehat{H}\Psi(\mathbf{q},t) = i\hbar\frac{\partial\Psi(\mathbf{q},t)}{\partial t}, \tag{4}$$

with Ψ the four-spinor.

For a free electron, the velocity operator is given by the Heisenberg equation

$$\widehat{v}_i = \frac{d\widehat{q}_i}{dt} = \frac{i}{\hbar}\left[\widehat{H},\widehat{q}_i\right] = \frac{\widehat{\Gamma}}{m_0}\widehat{p}_i + c\alpha_i, \tag{5}$$

where $\widehat{\Gamma} = -i\beta\gamma_5$. Expression (5) states that the electron possess two types of coordinates: an "external" one proportional to the usual velocity $\widehat{p}_i/m_0$ operator and an "internal" or "microscopic" [4] one given by $c\alpha_i$. In fact we can determine the position operator associated for the electron. To this end, we observe that

$$\left\{\widehat{H},\widehat{v}_i\right\} = 2\widehat{H}\widehat{v}_i + \left[\widehat{v}_i,\widehat{H}\right] = \frac{2E}{m_0}\widehat{p}_i, \tag{6}$$

where $E = +\sqrt{c^2\mathbf{p}^2 + m_0^2c^4}$. Thus

$$\frac{d\widehat{v}_i}{dt} = \frac{i}{\hbar}\left[\widehat{H},\widehat{v}_i\right] = \frac{2i}{\hbar}\widehat{H}\left(\widehat{v}_i - \frac{\widehat{H}^{-1}E}{m_0}\widehat{p}_i\right). \tag{7}$$

Let us define the operator

$$\widehat{\eta}_i \equiv \widehat{v}_i - \widehat{H}^{-1} E \frac{\widehat{p}_i}{m_0}. \tag{8}$$

Eq. (7) can be regarded as a differential equation $\widehat{v}_i$. Keeping in mind that $\widehat{p}_i$ and $\widehat{H}$ are constants of motion, we see that the $\widehat{\eta}_i$ satisfy the differential equations

$$\frac{d\widehat{\eta}_i}{dt} = \frac{2i}{\hbar} \widehat{H} \widehat{\eta}_i. \tag{9}$$

Solving for $\widehat{\eta}_i$ we get

$$\widehat{\eta}_i \left(t\right) = \exp\left(\frac{2i}{\hbar} \widehat{H} t\right) \widehat{\eta}_i \left(0\right). \tag{10}$$

As for the coordinate operator, the relation (10) can be integrated to yield

$$q_i \left(t\right) = q_i \left(0\right) + E \widehat{H}^{-1} \frac{\widehat{p}_i}{m_0} t - \frac{ic\hbar}{2} \widehat{H}^{-1} \exp\left(\frac{2i}{\hbar} \widehat{H} t\right) \widehat{\eta}_i \left(t\right).$$

For the electron at rest $\left(p_i = 0\right)$, we have that the operators

$$\begin{aligned} &\widehat{H}_0 = m_0 c^2 \beta, \qquad \widehat{S}_k = \frac{\hbar}{2} \Sigma_k \equiv -\frac{\hbar}{2} \alpha_i \alpha_j, \\ &\widehat{P}_i \equiv m_0 \left.\widehat{v}_i\right|_{\mathbf{p} \to \mathbf{0}} = m_0 c \alpha_i, \qquad \widehat{Q}_i \equiv \left.\widehat{q}_i\right|_{\mathbf{p} \to \mathbf{0}} = -\frac{i\hbar}{2 m_0 c} \beta \alpha_i, \end{aligned}$$

together with $i\gamma_5$, $\beta\gamma_5$ and $i\beta S_i$, form a $so(4,2)$ Lie algebra, corresponding to the dynamical symmetry of this system. Notice that this fact is naturally deduced by using the Dirac equation, i.e., from a relativistic point of view [4,5].

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